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Vibration Analysis of a Thin-Walled Cylindrical Shell Containing Fluid under Dynamic Loading

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Abstract

Liquid-filled cylindrical tanks are extensively used in industrial facilities, including petroleum, chemical, water supply, and nuclear power systems. These structures are frequently subjected to dynamic excitations caused by earthquakes, impact loads, and operational vibrations, which significantly influence their structural integrity and serviceability. The interaction between the contained liquid and the flexible tank wall alters the dynamic characteristics of the system by introducing additional inertia and fluid–structure coupling effects. Therefore, accurate prediction of the natural frequencies of partially filled tanks is essential for safe and reliable structural design. This study presents a dynamic free vibration analysis of a partially filled thin-walled cylindrical shell with one end fixed and the other end free. The fluid is assumed to be incompressible, inviscid, and irrotational, while the shell behavior is modeled according to thin shell theory. The governing equations are derived using the Rayleigh–Ritz energy method combined with Lagrange's equations. The fluid contribution is incorporated through the added-mass formulation obtained from the solution of the Laplace equation for the fluid domain. Numerical simulations are performed for both steel and concrete cylindrical tanks considering different liquid height ratios and shell geometric parameters. The results demonstrate that the natural frequencies increase as the liquid filling height decreases. Furthermore, increasing the shell height-to-radius ratio also leads to higher natural frequencies for both the first and second vibration modes. The proposed formulation provides an efficient analytical framework for evaluating the dynamic characteristics of liquid-filled cylindrical tanks and can serve as a practical tool for preliminary structural design and vibration assessment.

Keywords: Fluid–structure interaction, Thin cylindrical shell, Free vibration, Rayleigh–Ritz method, Added mass, Natural frequency.

1 | Introduction

Thin-walled cylindrical shells are among the most widely used structural components in petroleum, chemical, aerospace, nuclear, and water storage industries owing to their high strength-to-weight ratio and efficient

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load-carrying capacity [1–3]. In many engineering applications, these shells are employed as storage tanks or pressure vessels containing liquids, where they are continuously exposed to dynamic loading conditions such as earthquakes, blast loads, wind excitation, and operational vibrations [4], [5]. The dynamic response of these structures plays a crucial role in ensuring structural safety, operational reliability, and service life.

Unlike empty cylindrical shells, liquid-filled shells exhibit a complex Fluid–Structure Interaction (FSI), in which the motion of the contained fluid significantly influences the structural vibration characteristics. The hydrodynamic pressure generated by the fluid modifies the effective inertia of the system, resulting in changes in natural frequencies, vibration modes, and dynamic stability [6–8]. Consequently, neglecting the coupling effect between the shell and the fluid may lead to inaccurate predictions of structural behavior, particularly under dynamic loading conditions.

Theoretical developments in shell mechanics have established a solid foundation for investigating the behavior of thin-walled cylindrical structures. Classical elasticity and shell theories developed by Love [9], Calladine [10], Soedel [1], and Reddy [3] remain fundamental references for describing the deformation and vibration of thin shells. These formulations have subsequently been extended to include nonlinear behavior, large deformations, and complex loading conditions encountered in modern engineering applications [11], [12].

Several analytical and numerical approaches have been proposed to investigate the dynamic response of fluid-filled cylindrical shells. Among these methods, finite element formulations provide accurate solutions for complex geometries and loading conditions; however, they generally require significant computational resources [5]. Alternatively, analytical and semi-analytical techniques, particularly the Rayleigh–Ritz method, offer efficient solutions while preserving the physical interpretation of the coupled system [13]. Previous investigations have demonstrated that the added-mass effect of the contained liquid considerably reduces the natural frequencies of cylindrical shells, and this influence becomes more pronounced as the liquid filling ratio increases [14], [15–17].

Recent studies have further highlighted the importance of accurately modeling FSI in predicting the stability and dynamic performance of cylindrical shells. Feng et al. [6] developed a FSI model for laminated cylindrical shells and demonstrated that the interaction between the shell and the internal fluid substantially alters the dynamic stability boundaries. Similarly, Li et al. [12] investigated nonlinear deformation effects in thin-walled cylindrical shells, while Tănase [18] reviewed modern analytical approaches for evaluating the structural stability of isotropic and laminated cylindrical shells.

In the present study, the free vibration characteristics of a partially fluid-filled thin-walled cylindrical shell are investigated using the Rayleigh–Ritz energy method. The shell is modeled according to classical thin shell theory, whereas the contained liquid is assumed to be incompressible, inviscid, and irrotational. The fluid contribution is incorporated through an added-mass formulation derived from the velocity potential satisfying Laplace's equation. The effects of liquid filling ratio and shell geometric parameters on the natural frequencies are evaluated for both steel and concrete cylindrical shells. The proposed analytical formulation provides an efficient and reliable tool for the dynamic assessment of fluid-containing cylindrical shell structures.

2 | Literature Review

The dynamic behavior of cylindrical shells containing fluid has attracted considerable research interest because of its direct relevance to storage tanks, pressure vessels, and various industrial structures subjected to dynamic loading. Early theoretical developments were primarily based on classical shell mechanics established by Love [9], Calladine [10], and Soedel [1], which provided the mathematical framework for describing the deformation and vibration of thin-walled cylindrical shells. These formulations continue to serve as the basis for many contemporary analytical models.

The interaction between the shell and the enclosed fluid introduces additional complexity into the vibration problem because the hydrodynamic pressure generated by the moving liquid alters the effective inertia of the structural system. Morand and Ohayon [8] established one of the earliest comprehensive numerical formulations for FSI problems, while Ohayon and Soize [7] further developed analytical approaches for coupled acoustic–structural systems. Ibrahim [19] extensively investigated liquid sloshing phenomena and demonstrated their influence on the dynamic characteristics of liquid storage structures.

Analytical solutions based on energy principles have remained attractive due to their relatively low computational cost. Gonçalves and Batista [15] investigated the frequency response of partially liquid-filled cylindrical shells and demonstrated that increasing the liquid level significantly reduces the natural frequencies because of the added-mass effect. Amabili [14], [16] further developed analytical formulations for fluid-filled cylindrical shells and experimentally validated the influence of internal fluid on modal characteristics. These studies confirmed that fluid inertia dominates the vibration behavior of flexible cylindrical shells, particularly in lower vibration modes.

The Rayleigh–Ritz method has become one of the most widely used semi-analytical techniques for free vibration analysis of shell structures. Lee and Kwak [13] presented a comprehensive comparison of different shell theories using the Rayleigh–Ritz formulation and demonstrated that the method provides highly accurate predictions with substantially lower computational effort than full finite element analyses. Their work also emphasized the importance of selecting appropriate admissible displacement functions for achieving rapid convergence.

Several researchers have extended analytical models to include additional structural features. Kim et al. [17] investigated partially fluid-filled cylindrical shells reinforced with ring stiffeners and reported that structural stiffening considerably increases the natural frequencies while reducing vibration amplitudes. Yamaki [20] provided an extensive treatment of elastic stability and vibration of cylindrical shells, which remains one of the most comprehensive references in shell mechanics.

In recent years, research has increasingly focused on nonlinear behavior, advanced composite materials, and coupled stability analyses. Amabili [11] presented nonlinear formulations applicable to composite and biological shell structures, while Li et al. [12] examined the influence of nonlinear strain terms on large deformation responses of thin-walled cylindrical shells. Feng et al. [6] incorporated FSI into laminated shell models and demonstrated that coupling effects significantly modify stability limits and vibration characteristics. More recently, Tănase [18] reviewed modern analytical approaches for predicting the buckling behavior of isotropic and laminated cylindrical shells, highlighting the growing importance of efficient analytical techniques in engineering design.

Although high-fidelity finite element models have become increasingly popular for solving complex FSI problems [5], [12], analytical energy-based methods continue to offer important advantages for preliminary design and parametric investigations. Their computational efficiency, together with their ability to provide physical insight into the effects of fluid filling ratio, shell geometry, and boundary conditions, makes them particularly suitable for engineering applications. Motivated by these advantages, the present study employs the Rayleigh–Ritz method to investigate the free vibration characteristics of partially fluid-filled thin-walled cylindrical shells, with particular emphasis on the influence of liquid level and geometric parameters on the natural frequencies of steel and concrete shell structures.

3 | Research Methodology

In the present study, an analytical methodology based on the Rayleigh–Ritz energy approach is developed to investigate the free vibration characteristics of partially fluid-filled thin-walled cylindrical shells. The proposed formulation considers the coupled dynamic behavior of the shell and the contained liquid by incorporating the FSI effects through an equivalent added-mass formulation.

The cylindrical shell is modeled according to classical thin shell theory, assuming that the shell material behaves as a homogeneous, isotropic, and linearly elastic medium. The geometric parameters considered in the formulation include the shell radius, thickness, and length. The displacement field of the cylindrical shell is expressed using appropriate admissible functions that satisfy the imposed boundary conditions. These displacement functions are substituted into the strain energy and kinetic energy expressions of the shell to obtain the structural vibration equations.

The dynamic contribution of the internal liquid is introduced by assuming the fluid to be incompressible, inviscid, and irrotational. Under these assumptions, the fluid motion is described using a velocity potential function that satisfies Laplace's equation. The hydrodynamic pressure acting on the shell wall is obtained from the linearized Bernoulli equation, and the corresponding fluid kinetic energy is calculated. The effect of the contained liquid is then incorporated into the system through an added-mass matrix, which represents the additional inertia caused by fluid motion.

Using Hamilton's principle, the coupled fluid–structure vibration equation is derived by considering the total kinetic and potential energies of both the shell and the liquid. The Rayleigh–Ritz formulation is employed to transform the continuous vibration problem into a finite-dimensional eigenvalue problem. The natural frequencies of the fluid-filled cylindrical shell are obtained by solving the resulting characteristic equation.

A parametric study is subsequently performed to evaluate the influence of important variables on the vibration behavior of the coupled system. The investigated parameters include the liquid filling ratio, shell geometric dimensions, and material properties. Two commonly used structural materials, namely steel and concrete, are considered to compare their effects on the natural frequencies and dynamic characteristics of the cylindrical shells.

The accuracy of the proposed analytical model is assessed through comparison with previously published analytical and numerical results available in the literature. The obtained results are used to examine the role of FSI effects and to identify the dominant parameters controlling the vibration response of partially filled cylindrical shells.

This methodology provides an efficient analytical framework for predicting the dynamic behavior of fluid-containing cylindrical shell structures while maintaining the computational advantages of energy-based approaches.

2.1 | Description of the Investigated Fluid–Structure System

The investigated system, which consists of a liquid-containing tank, is shown in *Fig. 1*. The tank is considered as a vertical circular cylindrical shell with uniform thickness (h), radius (R), height (L), density (ρ), and Young's modulus (E). The tank is filled with liquid up to a height (H), and the density of the liquid is denoted by (ρ_l).

The liquid contained in the system is assumed to be incompressible and Newtonian. In the present study, a cylindrical coordinate system (r, θ, z) is used to define an arbitrary point within the coupled tank–liquid system. The displacement components in the axial, circumferential, and radial directions are denoted by (w), (v), and (u), respectively.

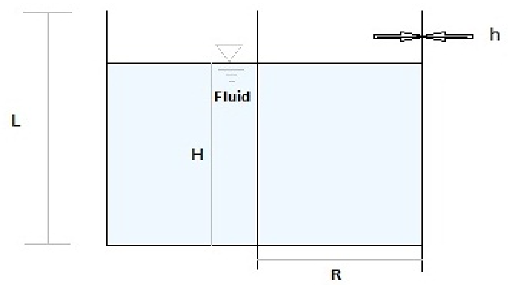

Fig. 1. Liquid-containing tank system.

$$u = \sum_{i=1}^{N_1} U_i(t)\chi_i(z)\cos\theta. \quad \textbf{(1)}$$

$$v = \sum_{i=1}^{N_2} V_i(t)\psi_i(z)\sin\theta. \quad \textbf{(2)}$$

$$w = \sum_{i=1}^{N_3} W_i(t)\psi_i(z)\sin\theta, \quad \textbf{(3)}$$

where $W_i(t), V_i(t), U_i(t)$ are the time-dependent displacement amplitudes associated with the functions $\chi_i(z)$ and $\psi_i(z)$, and N_3, N_2, N_1 represent the number of these functions used in the displacement approximation.

The governing differential equations are obtained using Lagrange's equation as follows.

$$\frac{d}{dt}\left(\frac{\partial(T_s + T_l)}{\partial \dot{q}_i}\right) - \frac{\partial(T_s + T_l)}{\partial q_i} + \frac{\partial S}{\partial q_i} = 0, \quad \textbf{(4)}$$

where T_s represents the kinetic energy of the tank, T_l denotes the kinetic energy of the liquid, S is the strain energy of the tank wall, and q_i represents the generalized coordinates.

The kinetic energy of the tank is obtained as follows.

$$T_s = \frac{\rho h}{2}\int_0^{2\pi}\int_0^{L} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) R\,dz\,d\theta, \quad \textbf{(5)}$$

where the dot notation (.) in the above equations indicates differentiation with respect to time.

The strain energy of the tank is obtained using the formulation provided by Chen et al. [21] as follows.

$$\begin{aligned} S = {} & \frac{E}{2(1-\nu^2)}\frac{h}{R}\int_0^{2\pi}\int_0^{L} [R^2 {u_z}^2 + (v_\theta + w)^2 \\ & + 2\nu R u_z (v_\theta + w) + \frac{1-\nu}{2}(u_\theta + R v_z)^2] dz\,d\theta \\ & + \frac{E}{24(1-\nu^2)}\frac{h^3}{R^3}\int_0^{2\pi}\int_0^{H} [R^4 {w_{zz}}^2 + (w_{\theta\theta} + w)^2 \\ & + \frac{1-\nu}{2}(R w_{z\theta} - u_\theta)^2 + \frac{3(1-\nu)}{2} R^2 (v_z + w_{z\theta})^2 \\ & + 2\nu R^2 w_{zz}(w_{\theta\theta} - v_\theta) - 2R^3 u_z w_{zz}] dz\,d\theta. \end{aligned} \quad \textbf{(6)}$$

The terms inside the first and second brackets represent the membrane and bending energies, respectively.

By substituting the expressions for S and T_s into *Eq. (4)* and neglecting T_l, a set of governing equations for the free vibration analysis of empty tanks is obtained. The resulting equations can be expressed in the following matrix form [18].

$$\begin{bmatrix} [A] & 0 & 0 \\ 0 & [B] & 0 \\ 0 & 0 & [C] \end{bmatrix} \begin{Bmatrix} \ddot{U} \\ \ddot{V} \\ \ddot{W} \end{Bmatrix} + \begin{bmatrix} [D] & [E] & [F] \\ [E] & [G] & [H] \\ [F] & [H] & [I] \end{bmatrix} \begin{Bmatrix} \{U\} \\ \{V\} \\ \{W\} \end{Bmatrix} = \{0\}, \quad \textbf{(7)}$$

where all the above matrices are square matrices, and $\{W\}$, $\{V\}$, and $\{U\}$ are column submatrices. The terms $U_i(i = 1,2, \dots, N_1)$, $V_i(i = 1,2, \dots, N_2)$, and $W_i(i = 1,2, \dots, N_3)$ represent the elements of the vectors $\{U\}$, $\{V\}$, and $\{W\}$, respectively.

The elements of matrices $[A]$ to $[I]$, along with the details of the differentiation process of *Eq. (7)*, can be found in the work of Rotter et al. [2]. The compact form of *Eq. (7)*, in the absence of damping effects, can be written as follows.

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}, \tag{8}$$

where $[M]$ is the mass matrix and $[K]$ is the stiffness matrix. The column matrix $\{q\}$ is defined as follows.

$$\{q\} = \begin{Bmatrix} U_1(t), U_2(t), \dots U_{N_1}(t) \\ , V_1(t), V_2(t), \dots V_{N_2}(t) \\ , W_1(t), W_2(t), \dots W_{N_3}(t) \end{Bmatrix}^T , \tag{9}$$

where the superscript T denotes the transpose of a row matrix.

The governing equations for a liquid-containing shell are obtained by adding the added liquid mass matrix to the mass matrix of the empty tank. The added liquid mass is only associated with the radial displacement, and the presence of the liquid has no effect on the stiffness matrix $[K]$.

If c_{ij} represents the elements of the submatrix $[C]$ in *Eqs. (3-7)*, then c_{ij}^* is defined as follows.

$$c_{ij}{}^* = c_{ij} + m_{ij}{}^l. \tag{10}$$

The quantity m_{ij}^l in the above equation is referred to as the added liquid mass.

The liquid is assumed to be inviscid and incompressible, and the velocity potential of the fluid satisfies the following Laplace equation.

$$\nabla^2\Phi = 0 \quad 0 \le r \le R\,, 0 \le \theta \le 2\pi\,, 0 \le z \le H. \tag{11}$$

The velocity component of the liquid at an arbitrary point in the (z)-direction can be expressed in terms of the scalar function (\Phi) as follows.

$$v_z = \frac{\partial \Phi}{\partial z}. \tag{12}$$

The boundary conditions for the liquid domain are defined as follows.

I. The vertical velocity of the liquid at the bottom of the tank must be zero, i.e.,

$$\frac{\partial \Phi}{\partial Z}|_z = 0. \tag{13}$$

II. The radial velocity of the liquid adjacent to the tank wall must be equal to the velocity of the tank wall, i.e.,

$$-\frac{\partial \Phi}{\partial r}|_{r=R} = \dot{w} = \sum_{i=1}^{N_3} \dot{W}(t)\psi_i(z)\cos\theta. \tag{14}$$

III. At the free surface, the pressure is assumed to be zero, i.e.,

$$\frac{\partial \Phi}{\partial t}|_{z=H} = 0. \tag{15}$$

By solving Laplace's equation using the method of separation of variables and applying the boundary *Conditions (13)* and *(15)*, the velocity potential function $\Phi(r, \theta, z, t)$ is obtained in the following form.

$$\Phi(r,\theta,z,t)=\sum_{n=1}^{\infty}C_n(t)I_1\left[\alpha_n\frac{r}{H}\right]\cos\left[\alpha_n\frac{z}{H}\right]\cos\theta, \tag{16}$$

where I_1 is the modified Bessel function of the first kind. $C_n(t)$ is a time-dependent function obtained from boundary *Condition (14)*, and:

$\alpha_n=\frac{(2n-1)\pi}{2}$ After determining $C_n(t)$ from *Eq. (14)* and substituting the result into *Eq. (16)*, the velocity potential function can be written as

$$\Phi=-\sum_{i=1}^{N_3}\dot{w}_i(t)\sum_{n=1}^{\infty}\frac{2H}{\alpha_n}\,\frac{I_1\left[\alpha_n\frac{r}{H}\right]}{\acute{I}_1\left[\alpha_n\frac{R}{H}\right]}\,d_{in}\cos\left(\alpha_n\frac{z}{H}\right), \tag{17}$$

where

$$d_{in}=\frac{1}{H}\int_0^H\psi_i(z)\cos\left(\alpha_n\frac{z}{H}\right)d\left(\frac{z}{H}\right). \tag{18}$$

The kinetic energy of an incompressible and inviscid liquid is calculated using the following relation proposed by Love [9].

$$T_l=\frac{\rho_l}{2}\int\int\Phi\,\frac{\partial\Phi}{\partial n}ds, \tag{19}$$

where Φ is the velocity potential function, $\frac{\partial\Phi}{\partial n}$ is the derivative of Φ in the direction normal to the liquid boundary, and (S) represents the surface integral over the wetted surface of the liquid domain.

Only the non-zero part of the surface integral in the kinetic energy equation is related to the tank wall, and the value of $\frac{\partial\Phi}{\partial n}$ is equal to $\frac{\partial\Phi}{\partial r}$. The contribution corresponding to the tank bottom is zero because the vertical velocity is assumed to be zero, and the contribution corresponding to the free surface is zero because the pressure is assumed to be zero. After eliminating the surface integrals, the liquid kinetic energy is obtained as

$$T_l=\frac{\rho_l}{2}\int_0^{2\pi}\int_0^{H}\left[\Phi\frac{\partial\Phi}{\partial r}\right]_{r=R}Rdzd\theta. \tag{20}$$

By substituting *Eq. (17)* into *Eq. (20)* and performing the required integrations, the final expression is obtained as

$$T_l=\frac{1}{2}\sum_{i=1}^{N_3}\sum_{j=1}^{N_3}m_{ij}{}^{l}\,\dot{w}_i\,\dot{w}_j, \tag{21}$$

where

$$m_{ij}{}^{l}=\left[\frac{H}{R}\sum_{n=1}^{\infty}\frac{2}{\alpha_n}\,\frac{I_1\left(\alpha_n\frac{R}{H}\right)}{I'_1\left(\alpha_n\frac{R}{H}\right)}\,d_{in}d_{jn}\right]m_l. \tag{22}$$

$m_l=p\rho_lR^2H$ is the total mass of the liquid. *Eq. (22)* represents the added liquid mass matrix in *Eq. (10)*.

By substituting the elements (C_{ij}) into the mass matrix ([M]), and assuming that the time-dependent vibration function has a sinusoidal form, *Eq. (8)* is transformed into the standard eigenvalue problem as follows.

$$[K]\{q\}=\omega^2[M]\{q\}, \tag{23}$$

where {q} is the eigenvector representing the vibration displacement amplitudes, and ω is the eigenvalue corresponding to the natural circular frequency of the coupled shell–liquid system.

3 | Numerical Results and Discussion

A flexible steel tank with Young's modulus $E = 30 \times 10^3$ psi, density $\rho = 490\text{Ib}$, height $L = 92\text{in}$, and thickness $h = 0/052$ in is considered. The tank contains a fluid with density $\rho_l = 62/4$ Ib./in at different liquid heights.

For different values of the ratios $\frac{H}{L}$ and $\frac{L}{R}$, the vibration frequencies of the tank in the first and second modes are calculated and presented in the following tables.

Table 1. Natural frequency of the steel tank (first mode).

$\frac{h}{R} = 0.001, \nu = 0.3, \frac{\rho_l}{\rho} = 0.127$				
$\frac{L}{R}$	$\frac{H}{L} = 1$	$\frac{H}{L} = 0.8$	$\frac{H}{L} = 0.6$	$\frac{H}{L} = 0.5$
0.3	6.3964	7.9944	10.6579	12.7889
0.5	6.4012	7.9981	10.6607	12.7913
0.75	6.4107	8.0057	10.6663	12.7959
1	6.4245	8.0164	10.6742	12.8024
1.5	6.4670	8.0487	10.6974	12.8214
2	6.5333	8.0977	10.7316	12.8490
2.5	6.6279	8.1666	10.7783	12.8862
3	6.7524	8.2581	10.8392	12.9339

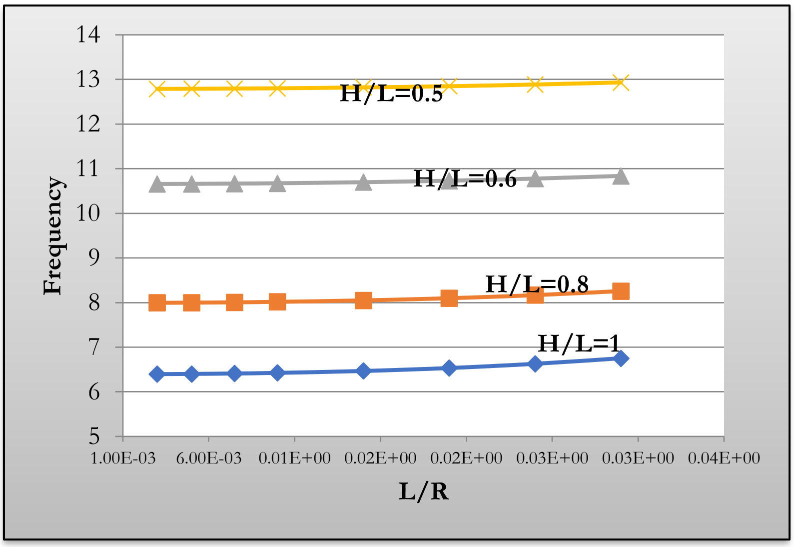

Fig. 2. Natural frequency of the steel tank (first mode).

Table 2. Natural frequency of the steel tank (second mode).

$\frac{h}{R}=0.001,\ \nu=0.3,\ \frac{\rho_l}{\rho}=0.127$				
$\frac{L}{R}$	$\frac{H}{L}=1$	$\frac{H}{L}=0.8$	$\frac{H}{L}=0.6$	$\frac{H}{L}=0.5$
0.3	6.4045	8.0008	10.6627	12.7929
0.5	6.4245	8.0164	10.6742	12.8024
0.75	6.4670	8.0487	10.6974	12.8214
1	6.5333	8.0977	10.7316	12.8490
1.5	6.7525	8.2582	10.8393	12.9339
2	7.0786	8.5129	11.0109	13.0665
2.5	7.4677	8.8483	11.2542	13.2557
3	7.8808	9.2335	11.5629	13.5048

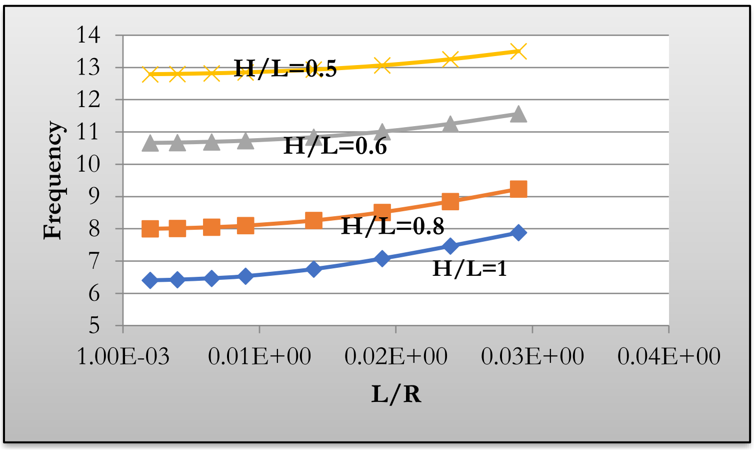

Fig. 3. Natural frequency of the steel tank (second mode).

By assuming a constant tank height (L), the above results for the first and second vibration modes of the liquid-containing steel tank were obtained for different values of the liquid height-to-tank height ratio (H/L) and the tank height-to-radius ratio (L/R). The results indicate that the natural frequency increases in both the first and second modes with decreasing H/L ratio and increasing L/R ratio.

Similar results were obtained for concrete tanks in the first and second vibration modes, which are presented in the following tables.

Table 3. Natural frequency of the concrete tank (first mode).

$\frac{h}{R}=0.01,\ \nu=0.15,\ \frac{\rho_l}{\rho}=0.5$				
$\frac{L}{R}$	$\frac{H}{L}=1$	$\frac{H}{L}=0.8$	$\frac{H}{L}=0.6$	$\frac{H}{L}=0.5$
0.3	10.3036	12.8767	17.1661	20.5976
0.5	10.3149	12.8857	17.1727	20.6034
0.75	10.3378	12.9035	17.1859	20.6143
1	10.3714	12.9295	17.2047	20.6298
1.5	10.4779	13.0090	17.2609	20.6755
2	10.6480	13.1330	17.3453	20.7428
2.5	10.8890	13.3100	17.4631	20.8350
3	11.1938	13.5439	17.6191	20.9556

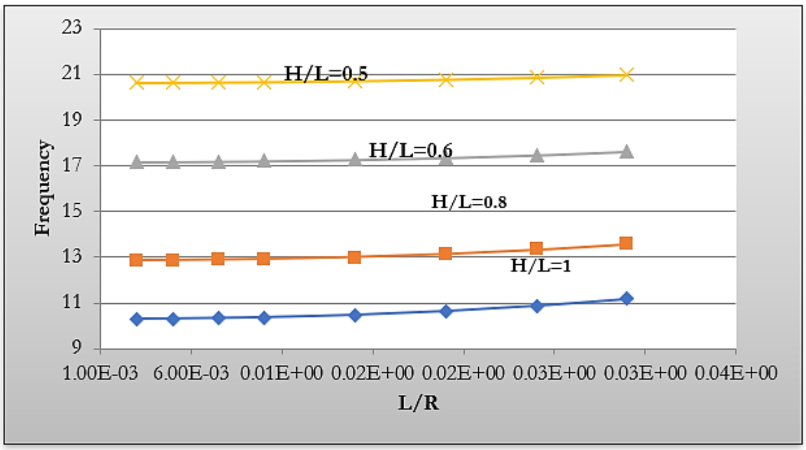

Fig. 4. Natural frequency of the concrete tank (first mode).

Table 4. Natural frequency of the concrete tank (second mode).

$\frac{h}{R}=0.01,\ \nu=0.15,\ \frac{\rho_1}{\rho}=0.5$				
$\frac{L}{R}$	$\frac{H}{L}=1$	$\frac{H}{L}=0.8$	$\frac{H}{L}=0.6$	$\frac{H}{L}=0.5$
0.3	10.3328	12.8919	17.1661	20.6069
0.5	10.3714	12.9295	17.1727	20.6298
0.75	10.4778	13.0090	17.1859	20.6755
1	10.6480	13.1330	17.2047	20.7428
1.5	11.1941	13.5442	17.2609	20.9557
2	11.9275	14.1627	17.3453	21.2960
2.5	12.7290	14.9094	17.4631	21.7780
3	13.5364	15.7082	17.6191	22.3875

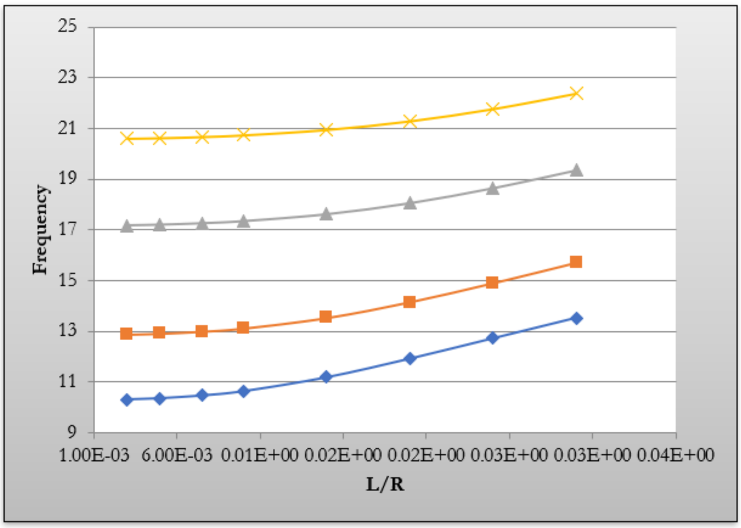

Fig. 5. Natural frequency of the concrete tank (second mode).

4 | Conclusion

Based on the analyses performed and the obtained results, the main conclusions of this study can be summarized as follows.

I. Increasing the ratio of tank height to radius, as well as decreasing the ratio of liquid height to tank height, results in an increase in the natural frequency of the liquid-containing tank system for both steel and concrete tanks in the first and second vibration modes. For example, for the liquid-containing steel tank in the first mode, when (L/R=0.3) and for (H/L=1, 0.8, 0.6,) and (0.5), the natural frequencies are obtained as 6.3964, 7.9944, 10.6579, and 12.7889, respectively. In the second mode, the corresponding values are 6.4045, 8.008, 10.6627, and 12.7929, respectively.

For (L/R=3) and the same values of (H/L), the natural frequencies of the liquid-containing steel tank are obtained as 6.7524, 8.2581, 10.8392, and 12.9339 in the first mode, and 7.8808, 9.2335, 11.5629, and 13.5048 in the second mode, respectively.

II. For the liquid-containing concrete tank, the natural frequencies in the first mode for (L/R=0.3) and (H/L=1, 0.8, 0.6,) and (0.5) are obtained as 10.3036, 12.8767, 17.1661, and 20.5976, respectively. The corresponding values in the second mode are 10.3228, 12.8919, 17.1774, and 20.6069, respectively.

For (L/R=3) and the same (H/L) ratios, the natural frequencies of the liquid-containing concrete tank are obtained as 11.1938, 13.5439, 17.6191, and 20.9556 in the first mode, and 13.5364, 15.7082, 19.3670, and 22.3875 in the second mode, respectively.

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